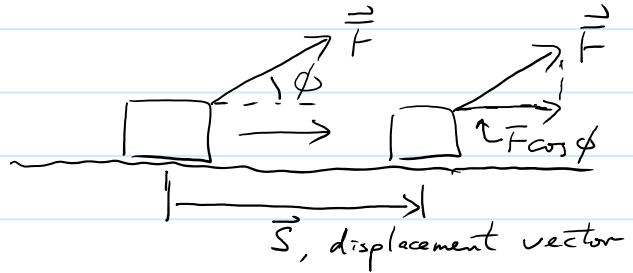


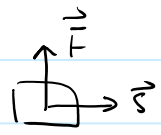
Lecture 4. Work and Kinetic Energy

Work (energy transfer to an object due to a force)

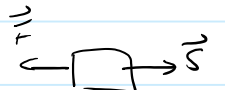
Work done by a force

$$W = F \cdot s \cdot \cos \phi$$



for $\phi = 90^\circ$,  $\Rightarrow W = 0$ $\therefore$ No changes in energy (kinetic)
 $\therefore W = 0$

for $\phi = 0^\circ$,  $\Rightarrow W = F \cdot s$, $a > 0 \therefore$ speed increases
 $\therefore W > 0$

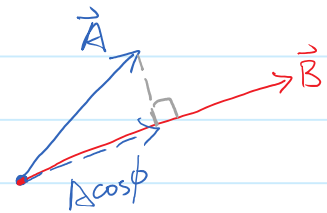
for $\phi = 180^\circ$,  $\Rightarrow W = -F \cdot s$, $a < 0 \therefore$ speed decreases
 $\therefore W < 0$

S.I. unit of work : 1 Joule = 1 J = 1 N.m.

Dot Product.

we write: $W = \vec{F} \cdot \vec{s} = |\vec{F}| \cdot |\vec{s}| \cdot \cos \phi$

$$\vec{A} \cdot \vec{B} = A B \cos \phi = \vec{B} \cdot \vec{A}$$



if $\phi = 0$, $\vec{A} \cdot \vec{B} = A \cdot B$

if $\phi = 90^\circ$, $\vec{A} \cdot \vec{B} = 0$

$$\Rightarrow \begin{cases} \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1 \\ \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0 \end{cases}$$

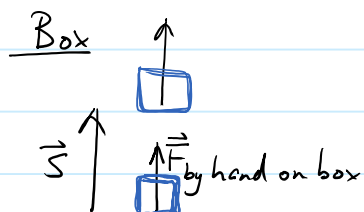
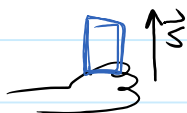
$$\vec{A} = A_x \hat{i} + A_y \hat{j}, \quad A_x = \vec{A} \cdot \hat{i}, \quad A_y = \vec{A} \cdot \hat{j}$$

$$\vec{A} \cdot \vec{B} = (A_x \hat{i} + A_y \hat{j}) \cdot (B_x \hat{i} + B_y \hat{j}) = A_x B_x \hat{i} \cdot \hat{i} + A_x B_y \hat{i} \cdot \hat{j} + A_y B_x \hat{j} \cdot \hat{i} + A_y B_y \hat{j} \cdot \hat{j}$$

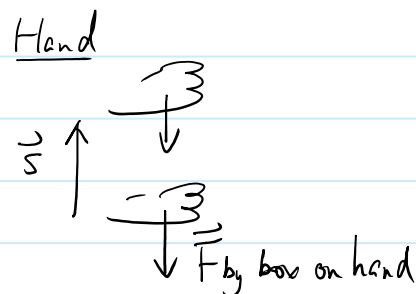
$$\boxed{\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y}$$

$$W = \vec{F} \cdot \vec{s} = F_x s_x + F_y s_y + F_z s_z \quad \text{scalar product.}$$

e.g. Hand lifts a box



$$W_{\text{on box}} = \vec{F}_{\text{by hand on box}} \cdot \vec{s} > 0 \quad (\text{same direction})$$



$$W_{\text{on hand}} = \vec{F}_{\text{by box on hand}} \cdot \vec{s} < 0 \quad (\text{opp. direction})$$

$$\text{But } \underline{W_{\text{on box}}} = \vec{F}_{\text{by hand on box}} \cdot \vec{s} = -\vec{F}_{\text{by box on hand}} \cdot \vec{s} = -\underline{W_{\text{on hand}}}$$

action - reaction pair.
(Newton's 3rd Law)

Energy transfer to the box (+ve) = Energy transfer away from the hand (-ve)

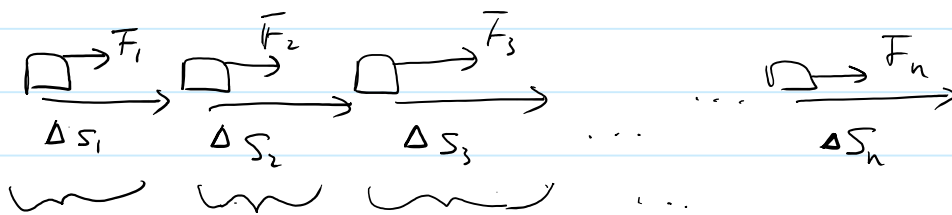
$$\text{Multiple Forces: } W_{\text{tot}} = \sum_i \vec{F}_i \cdot \vec{s} = \sum_i W_i$$

Recall: Work-Energy Theorem $K = \frac{1}{2} m v^2$

$$\Delta K = W_{\text{tot}} \Rightarrow W_{\text{tot}} = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2$$

Constant force $\Rightarrow$ constant $a \Rightarrow 2as = v_f^2 - v_i^2$
 $\Rightarrow \underline{ma \cdot s} = \frac{1}{2} m (v_f^2 - v_i^2)$
 $W_{\text{tot}} = \sum \vec{F}_i \cdot \vec{s} = \frac{1}{2} m (v_f^2 - v_i^2) \quad \checkmark$

Non-constant force



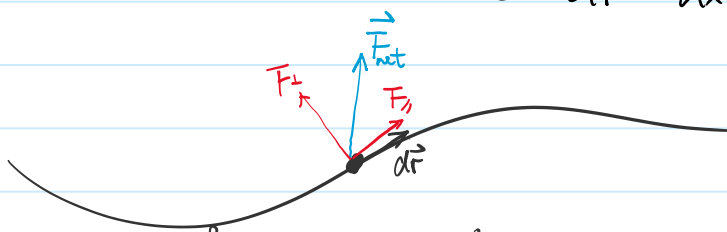
$$W = \Delta W_1 + \Delta W_2 + \Delta W_3 + \dots + \Delta W_n$$

⇒ Consider $F = F(x)$ function of position.

$$W = \sum_i \Delta W_i = \sum_i \vec{F}_i \cdot \Delta \vec{s}_i \rightarrow \int_{x_i}^{x_f} F(x) \cdot dx$$

in 3D
$$W = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r} = \int_{x_i}^{x_f} F_x \cdot dx + \int_{y_i}^{y_f} F_y \cdot dy + \int_{z_i}^{z_f} F_z \cdot dz$$

where $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$



$$W_{\text{tot}} = \int_i^f \vec{F}_{\text{net}} \cdot d\vec{r} = \int_i^f F_{\parallel} \cdot dr$$

$$= \int m a_{\parallel} \cdot dr$$

$$= \int m \frac{dv}{dt} \cdot dr \quad \because dr = \frac{dr}{dt} dt$$

$$= \int m \frac{dv}{dt} v dt = \int v dv$$

$$= \int_{v_i}^{v_f} m v dv$$

$$= \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2$$

$$W_{\text{tot}} = K_f - K_i \quad \text{where } K = \frac{1}{2} m v^2$$

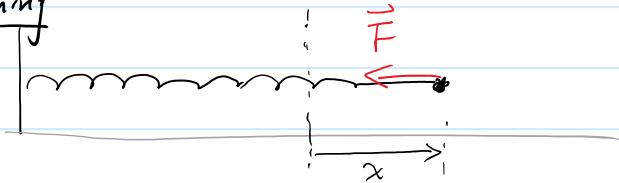
Hooke's Law.

- Restoring force e.g. tension of a spring.



$x=0$
natural position / equilibrium position.

stretching

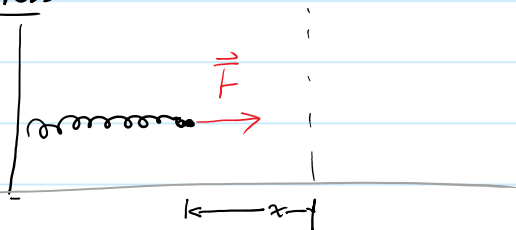


Force by the spring ($\vec{F} = F_x \hat{i}$)

$$F_x = -kx. \quad (x > 0)$$

to the left

compress



but $x < 0$

$$F_x = k|x| = -kx$$

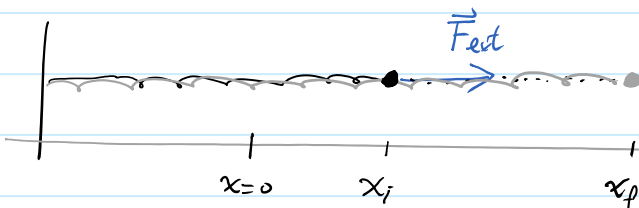
to the right.

$$\vec{F}_{\text{spring}} = -\frac{k}{x} \hat{i}$$

displacement from the natural position.

force constant / spring constant
(Force per unit length.)

Work by an external force on a spring (at equilibrium $\vec{a} = \vec{0}$)



$$\vec{F}_{\text{ext}} = -\vec{F}_{\text{spring}}$$

when stretching slowly.

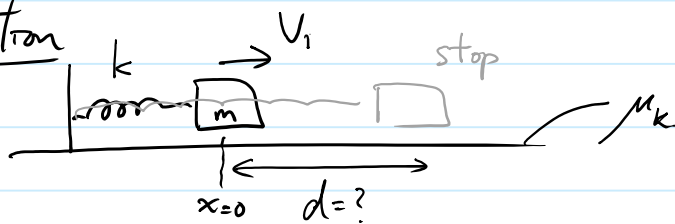
$$W_{\text{on spring}} = \int_{x_i}^{x_f} \vec{F}_{\text{ext}} \cdot d\vec{x} \hat{i} = - \int_{x_i}^{x_f} \vec{F}_{\text{spring}} \cdot d\vec{x} \hat{i} = k \int_{x_i}^{x_f} x dx$$

$$= \frac{1}{2} k x_f^2 - \frac{1}{2} k x_i^2$$

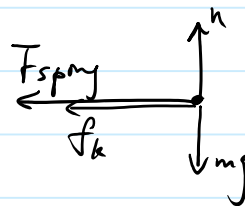
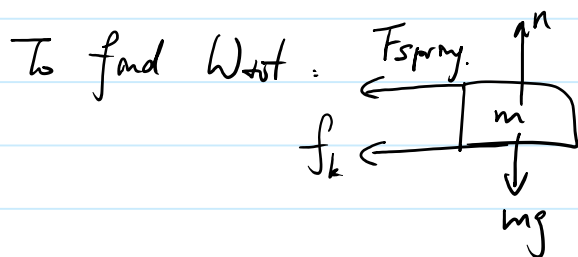
stretching: $x_f > x_i$, $W > 0$ work done on spring.

releasing: $x_f < x_i$, $W < 0$, work done by spring.

Application



$$\Delta K = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2 = 0 - \frac{1}{2} m v_i^2$$



$$f_k = \mu_k mg$$

$$W_{\text{tot}} = W_f + W_{\text{by spring}}$$

$$W_f = \vec{f}_k \cdot \vec{d} = -\mu_k mg d$$

$$W_{\text{by spring}} = -\left(\frac{1}{2} k x_f^2 - \frac{1}{2} k x_i^2\right) = -\frac{1}{2} k d^2 - 0$$

$$W_{\text{tot}} = \Delta K \text{ gives}$$

$$-\frac{1}{2} k d^2 - \mu_k mg d = -\frac{1}{2} m v_i^2$$

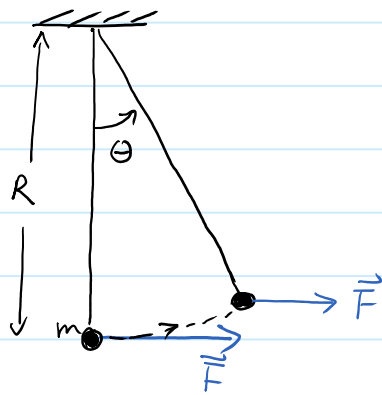
$$d = \frac{-\mu_k mg}{k} \pm \sqrt{\left(\frac{\mu_k mg}{k}\right)^2 + \frac{m v_i^2}{k}}$$

$$W_{\text{by spring}} = -W_{\text{on spring}}$$

$$\text{or } = \int F_{\text{spring}} \cdot dx$$

$$= \int_{x_i}^{x_f} (-kx) dx$$

Example in 2D. Swing



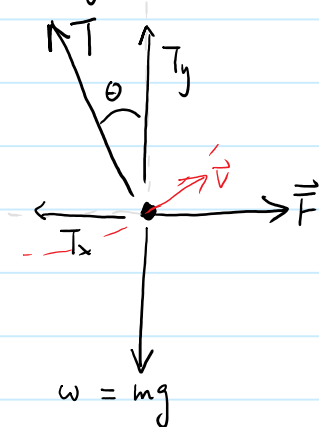
situation: apply a horizontal force $\vec{F}$ to push the swing up from $\theta=0$ to $\theta=\theta_0$.

Find the work done by each force acting on the object.

Assumption: $\vec{F}$ is applied such that the objects move up slowly at a constant speed (at equilibrium any time)

$$\Rightarrow \sum \vec{F} = \vec{0} \Rightarrow \Delta K = 0$$

At any angle θ ,



$$\vec{a} = \vec{s}$$

$$\begin{aligned} x: & F = T_x = T \sin \theta \\ y: & W = T_y = T \cos \theta \end{aligned} \quad \left. \vphantom{\begin{aligned} x: \\ y: \end{aligned}} \right\} \frac{F}{W} = \tan \theta$$

$$\text{or } F = W \tan \theta = mg \tan \theta$$

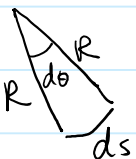
$$W_T = \int \vec{T} \cdot d\vec{s} = 0$$

$\because \vec{T} \perp d\vec{s}$

$$W_F = \int dW_F$$

$$\begin{aligned} dW_F &= \vec{F} \cdot d\vec{s} \\ &= F \cdot ds \cdot \cos \theta \\ &= mg \tan \theta \cdot R d\theta \cos \theta \\ &= mg R \sin \theta d\theta \end{aligned}$$

$$W_F = \int_0^{\theta_0} mg R \sin \theta d\theta = mg R (1 - \cos \theta_0)$$



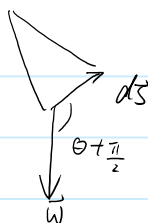
$$|d\vec{s}| = R d\theta$$

$d\vec{s}$
infinitesimal
displacement

$$W_w = \int_0^{\theta_0} mg R d\theta \underbrace{\cos(\theta + \frac{\pi}{2})}_{-\sin\theta}$$

$$= -mgR \int_0^{\theta_0} \sin\theta d\theta$$

$$= -mgR (-\cos\theta) \Big|_0^{\theta_0} = mgR (\cos\theta_0 - 1)$$



check: $W_{tot} = W_T + W_F + W_w = \Delta K$

$$\Delta K = 0 \quad \because \vec{a} \sim \vec{0} \text{ throughout the motion.}$$

$$W_{tot} = W_T + W_F + W_w$$

$$= 0 + mgR(1 - \cos\theta_0) + mgR(\cos\theta_0 - 1) = 0$$

$$\Rightarrow W_{tot} = \Delta K \quad \checkmark$$

Power $\frac{\text{Work}}{\text{time}} = \frac{\Delta W}{\Delta t}$

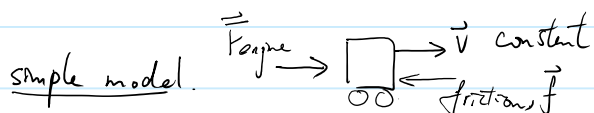
$$P = \frac{dW}{dt} \quad \text{unit: 1 Watt} = 1 \text{ J/s.}, \quad 1 \text{ horse power} = 746 \text{ W}$$

$$P = \frac{dW}{dt} = \frac{d}{dt} \int \vec{F} \cdot d\vec{s} = \frac{d}{dt} \int \vec{F} \cdot \frac{d\vec{s}}{dt} dt = \frac{d}{dt} \int \vec{F} \cdot \vec{v} dt = \vec{F} \cdot \vec{v}$$

$$\boxed{P = \vec{F} \cdot \vec{v}}$$

example. car moving at constant speed v
given power of the car engine.

find frictional force acting on the car.



$$\vec{F}_{\text{engine}} - \vec{f} = 0 \quad \because \vec{a} = \vec{0}$$

$$P = \vec{F}_{\text{engine}} \cdot \vec{v} = F_{\text{engine}} v = f \cdot v$$

$$\Rightarrow f = \frac{P}{v}$$

Alternative proof on work-energy theorem.

Proof: work-energy theorem with non-constant force. (1D)

$$\begin{aligned} W_{tot} &= \int_{x_i}^{x_f} \Sigma \vec{F}_{ix} dx \\ &= \int_{x_i}^{x_f} m a_x \cdot dx \\ &= m \int_{x_i}^{x_f} v_x \frac{dv_x}{dx} dx \\ &= m \int_{v_i}^{v_f} v_x dv_x \\ &= m \left(\frac{1}{2} v_x^2 \right) \Big|_{v_i}^{v_f} \\ W_{tot} &= \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2 = \Delta K. \end{aligned}$$

$\therefore a_x = \frac{dv_x}{dt} = \frac{dx}{dt} \cdot \frac{dv_x}{dx} = v_x \frac{dv_x}{dx}$

One can work out a proof in 3D by

$$\int \vec{F} \cdot d\vec{s} = \int F_x dx + \int F_y dy + \int F_z dz$$